Problem 5097. Let $p \geq 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor}$$

where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

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Let us denote by $f(n)$ and $S(n)$ the terms and the partial sums of the given series respectively, i.e.

$$f(n) = \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor} \qquad , \qquad S(n) = \sum_{k=1}^{n} \frac{(-1)^k}{\left\lfloor \sqrt[p]{k} \right\rfloor}$$

Note that $f(n)$ is a decreasing sequence and $f(n) \to 0$ as $n \to +\infty$, hence, our series is convergent by virtue of Leibnitz's criterion. Thus, in order to determine the requested sum, we can consider the subsequence $T(n) = S((2n)^p)$.

Clearly we have:

$$\begin{aligned} T(n) &= f\left(2^p - 1\right) + f\left(2^p\right) + f\left(4^p - 1\right) + f\left(4^p\right) + \cdots + f\left((2n)^p - 1\right) + f\left((2n)^p\right) = \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots - \frac{1}{2n-1} + \frac{1}{2n} \end{aligned}$$

since for all $n \in \mathbb{N}$ and $i \in 1, 3, 5, \ldots, (2n)^p - 3$ we have

$$f\left((n-1)^p + i\right) + f\left((n-1)^p + i + 1\right) = 0$$

Thus $T(n)$ is exactly the partial sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Therefore, thanks to a well known result, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor} = -\log 2$$

and the proof is finished. $\square$